THERMAL STESSES IN A CONTINUOUS PLATE INGOT

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The stress field in a plate during solidification and cooling from an ingot is analyzed in the elastic-body approximation.

1. The state of stress in a continuous ingot is largely determined by the thermal stresses which appear during the forming process. An analysis of these stresses is tied to the general analysis of thermal stresses in a medium during its solidification and to a number of analogous specific situations: growth of crystals, hardening of polymers, crystallization of welding seams, solidification of molten rocks, freezing of soils, etc. The physical aspect of this problem has not been explored thoroughly enough. Mathematically, however, the problem has been dealt with in a number of studies [1-12]. Every analysis so far was based on the theory of uncoupled thermal deformation and applied to one or another rheological model of bodies with simple geometries.

The first studies based on the elastic-body approximation were made by T. Hirone [1, 2], who analyzed stresses in a freely solidifying cylinder and sphere. The elastic-body approximation was subsequently used also in [3-7].

The general approach to the calculation of thermoelastic stresses in solidifying bodies was shown in [6], where the time derivatives of both the strain and the stress tensor were assumed to satisfy the conventional defining equation of thermoelasticity and all components of the stress tensor were assumed equal to zero at the interphase boundary. The latter condition was also stipulated by T. Hirone, V. L. Indenbom, and G. Reeder. A similar approach in the elastic—plastic approximation was taken earlier by B. Boley and I. Weiner [8], who calculated the thermal stresses in an infinitely large solidifying plate under a Neumann temperature distribution in the solid phase. A solution in closed form was obtained assuming, in addition, a constant ratio of the depth of plastic zones to the total depth of solidifying mass. The stresses were found then to be tensile within the interphase boundary zone and compressive at the cooled surface. K. Tien and V. Kaump analyzed thermal stresses in a solidifying metallic plate ingot on the basis of an elastic-beam model. They suggested that during one-dimensional solidification there appears a thin elastic solidifying layer across the entire plate width, equivalent to a beam either freely supported or fixed at both ends. It was noted, moreover, that the stresses at the interphase boundary changed from compressive to tensile in the first version and remained compressive throughout in the second version. M. Ya. Brobman and E. V. Surin [9] used the elastic-body model for analyzing the one-dimensional symmetrical problem of an infinitely large plate solidifying under a linear temperature distribution in the building up solid phase. The character of the stress distribution was analogous here to that established in [8]. The method of reducing the problem of stresses in an elastically solidifying body to conventional thermoelasticity problems was developed by E. A. Iodko [11, 12]. In [12] he considered the solidification of a plate and of an infinitely long cylinder, with the interphase boundary assumed moving either linearly or parabolically with time. In the first case the region adjacent to the solidification front was found to be under compression and the region near the cooling surface under tension. In the second case the stress distribution was found to be reverse. In this study here the authors will consider thermal stresses in an elastically solidifying plate under a Stefan temperature distribution in the solid phase. We will analyze the dependence of the stress field in a plate on the solidification rate, and also the set of thermophysical properties which characterize this kind of situation. Considerable attention will be paid to an evaluation of stresses appearing in a plate

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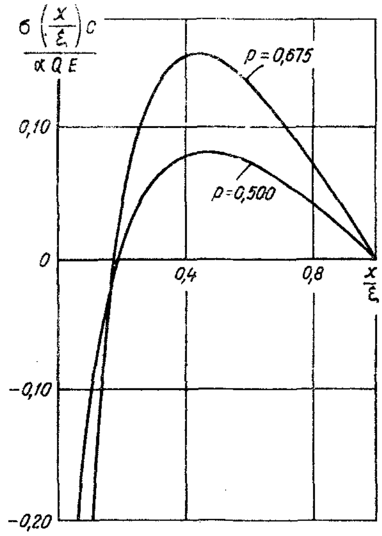

Fig. 1. Distribution of thermal stresses in a solidifying plate.

at the end of the solidification process under various modes of subsequent cooling. The residual stresses will also be determined.

2. We consider the classical Stefan model of a solidfying medium [13] existing in two states: liquid and solid. Initially the medium is completely liquid at a temperature T_0. At any subsequent instant of time $t > 0$ the surface temperature of the medium remains constant $T_c < T_0$. At every instant of time the interphase boundary is in a definite position and its temperature is T_S.

We assume that the building up solid phase conforms to the elastic-body model. We isolate within the liquid region an arbitrary material point with zero stress. Let that point find itself on the interphase boundary at the time $t = \tau$. This means that elastic stresses will from now on appear at the point and, varying at the rate σ_{ij}, they will at time $t > \tau$ reach the level

$$\sigma_{ij} = \int_{\tau}^{t} \dot{\sigma}_{ij} dt. \tag{1}$$

Inasmuch as the stress at the given point starting at time $t = \tau$ satisfies Hooke's Law

$$\sigma_{ij} = 2\mu_0 \varepsilon_{ij} + \lambda_0 \Theta U_{ij} + k\alpha (T_S - T) U_{ij} \tag{2}$$

therefore, differentiating (2) with respect to time and inserting the resulting expression into (1) will yield Hook's defining equation for a solidifying medium:

$$\sigma_{ij} = 2\mu_0 \int_{\tau}^{t} \dot{\varepsilon}_{ij} dt + \lambda_0 U_{ij} \int_{\tau}^{t} \dot{\Theta}\, dt - k\alpha \int_{\tau}^{t} \dot{T} dt. \tag{3}$$

Using these concepts now, we will formulate the problem in terms of the theory of uncoupled thermoelasticity and corresponding equations of motion, defining equations, initial conditions, and boundary conditions. It will be assumed here that the only nonzero components of both the stress and the strain tensor are $\sigma_{yy} = \sigma_{zz} = \sigma(x, t)$, $\varepsilon_{yy} = \varepsilon_{zz} = \varepsilon(x, t)$.

$$\partial_t T_1 = a_1 \partial_x^2 T_1, \quad \xi < x < \infty; \tag{4}$$

$$\partial_t T = a \partial_x^2 T, \quad 0 < x < \xi; \tag{5}$$

$$T(x,\ 0) = T_0; \quad T(\xi,\ t) = T_S, \tag{6}$$

$$\lambda \partial_x T - \lambda_1 \partial_x T_1 = \rho Q_0 \partial_t \xi, \quad x = \xi, \tag{7}$$

$$\rho \partial_t \dot{U} = \partial_x \sigma(x,\ t), \quad 0 < x < \xi, \tag{8}$$

$$\sigma(x,\ t) = \int_{\tau}^{t} \{E\dot{\varepsilon}(x,\ t) - \alpha E\dot{T}(x,\ t)\}\, dt, \tag{9}$$

$$\int_{0}^{\xi} \sigma(x,\ t)\, dx = f(t). \tag{10}$$

3. Problem (8)-(10) is solved in the quasistatic mode, with the independently determined temperature field as the load function. The temperature distribution in the solid phase corresponds to the solution to the Stefan problem (4)-(7):

$$T = T_c + \frac{Q}{c} \sqrt{\pi}\, p \exp(p^2) \operatorname{erf}\left(p \frac{x}{\xi}\right), \tag{11}$$

$$\xi = p\sqrt{4at}. \tag{12}$$

Here $Q = Q_0 + c_1(T_0 - T_S)$, i.e., the superheat of the liquid is included in the equivalent heat of solidification. The solution to the problem ought to be sought in the form of a function which satisfies the condition of compatibility with regard to strains ($\partial_{kl}^2 \varepsilon_{ij} = 0$). In our case this condition is satisfied identically and, therefore, $\varepsilon(x, t) = \varepsilon(t)$. With the aid of this relation, and considering the plate to be free of external forces $f(t) = 0$, we find that the balance of forces (10) and the temperature field (11) yield

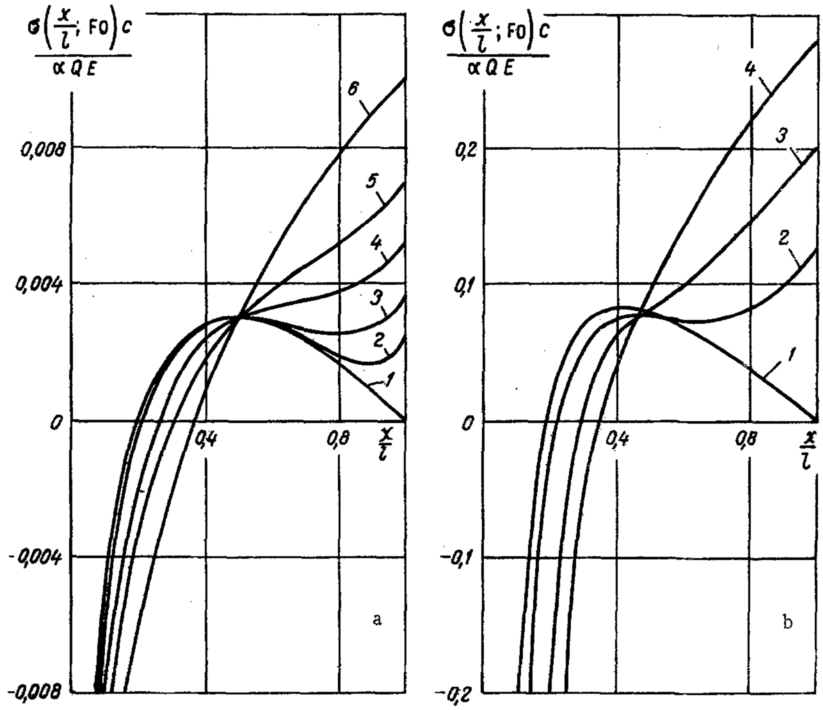

Fig. 2. Distribution of thermal stresses across the section of a cooling plate, from the instant solidification has been completed: (a) p = 0.1 and Bi = 0, Fo = 0 (1), 0.01 (2), 0.025 (3), 0.05 (4), 0.10 (5), ∞ (6), (b) p = 0.5 and Bi = 0.5, N = 3.66, Fo = 0 (1), 1.0 (2), 2.5 (3), ∞ (4).

$$\Psi(t) = \frac{\alpha Q}{2c}\left[1 - \exp(p^2)\right]\frac{1}{t}. \tag{13}$$

The prerequisite distribution of thermal stresses in a solidifying plate is found from (9) and (13):

$$\frac{\sigma\left(\frac{x}{\xi};p\right)c}{E\alpha Q} = \sqrt{\pi}\,p\exp(p^2)\left\{\operatorname{erf}(p) - \operatorname{erf}\left(\frac{x}{\xi}p\right)\right\} + \left\{\exp(p^2) - 1\right\}\ln\frac{x}{\xi}. \tag{14}$$

Expression (14) indicates that the generated stresses are proportional to the quantity $E\alpha Q/c$, which characterizes macroscopically the forces of interatomic bonds. It is to be noted that the dimensionless quantity $Q_0\alpha/c$, with the mean-integral values of α and c over the temperature range from room to melting, is the same for several metals and equal to $0.9\cdot10^{-2}$. This can be easily explained in the case of simple monoatomic substances, such as metals, in the light of Debye's theory of heat capacity. Accordingly, the level of stresses generated during the solidification of various metals is determined only by the magnitude of the elastisticity modulus E and of the solidification parameter p which characterizes the process conditions and the thermophysical properties of the medium.

In Fig. 1 is shown the distribution of thermal stresses $[\sigma(x/\xi)c/\alpha EQ] = f(x/\xi)$ in a solidifying plate, which has been calculated according to Eq. (14) for various process conditions (p = 0.5 and 0.675).

It follows from Fig. 1 that the solid region adjacent to the solidification front is under tension, while the region adjacent to the plate surface is under compression. The maximum tensile stress increases fast as p increases. At the same time, at any value of p, near the cooled surface there appears a plastic region. When p is small ($p < 0.1$), expression (14) can be simplified by retaining only the first terms in p of the exponential expansion and of the Gauss function:

$$\frac{\sigma\left(\frac{x}{\xi},p\right)c}{\alpha EQ} = \frac{p^2}{1-p^2}\left(\ln\frac{x}{\xi} - 2\frac{x}{\xi} + 2\right). \tag{15}$$

4. After the solidification process has been completed, a continuous ingot continues to cool down within the zone of secondary cooling and in air. At the end of solidification ($\xi = l$) the temperature field of the ingot is, according to (11),

$$T\left(\frac{x}{l}, p\right) = T_c + \frac{Q}{c}\sqrt{\pi}\exp(p^2)\operatorname{erf}\left(p\frac{x}{l}\right). \tag{16}$$

Here $2l$ is the plate thickness. The distribution of thermal stresses at that instant of time is, according to (14),

$$\frac{\sigma\left(\frac{x}{l}, p\right)c}{\alpha EQ} = \sqrt{\pi}\, p\exp(p^2)\left\{\operatorname{erf}(p) - \operatorname{erf}\left(\frac{x}{l}p\right)\right\} + \left\{\exp(p^2) - 1\right\}\ln\frac{x}{l}. \tag{17}$$

The subsequent changes in the stresses can be determined from the known temperature field of the cooling plate. For this purpose, we solve the problem of symmetrical cooling of an infinitely large plate, assuming that the heat transfer at its boundary follows Newton's Law:

$$\partial_x T(l, t) = -\frac{\alpha_T}{\lambda}\{T(l, t) - T_a\}.$$

Here T_a is the temperature of the cooling medium and α_T is the coefficient of heat transfer at the plate surface.

As the initial temperature distribution we consider (16), then the solution to the given thermal problem with its general solution in [13] will be

$$T\left(\frac{x}{l}, \mathrm{Fo}\right) - T_a = \sum_{n=1}^{\infty}\frac{4\mu_n}{2\mu_n + \sin 2\mu_n} \times \left\{\int_0^1\left[T_c - T_a + \frac{Q}{c}\sqrt{\pi}\exp(p^2)\operatorname{erf}\left(p\frac{x}{l}\right)\right]\cos\mu_n\left(1 - \frac{x}{l}\right)d\left(\frac{x}{l}\right)\right\} \times \cos\mu_n\left(1 - \frac{x}{l}\right)\exp\left(-\mu_n^2\mathrm{Fo}\right).$$

Here μ_n are the roots of the characteristic equation $\mu_n = \mathrm{Bi}\cdot\cot\mu_n$. Reducing the integral on the right-hand side and then simplifying, we obtain

$$c\frac{T\left(\frac{x}{l}, \mathrm{Fo}\right) - T}{Q} = \sum_{n=1}^{\infty} A_n\cos\mu_n\left(1 - \frac{x}{l}\right)\exp\left(-\mu_n^2\mathrm{Fo}\right), \tag{18}$$

$$A_n = Nm_n - \sqrt{\pi}\, p\exp(p^2)\operatorname{erf}(p)\, m_n + h_n\sqrt{\pi}\, p\exp(p^2) \times \int_0^1 \exp\left(\frac{x}{l}p\right)\cos\mu_n\left(1 - \frac{x}{l}\right)d\left(\frac{x}{l}\right),$$

$$m_n = \frac{4\sin\mu_n}{2\mu_n + \sin 2\mu_n};\quad h_n = \frac{4\mu_n}{2\mu_n + \sin 2\mu_n};\quad N = \frac{c(T_s - T_a)}{Q}.$$

With the aid of the temperature field we can then find the distribution of stresses generated in the plate starting from the time solidification has been completed. The mean-over-the-section plate temperature at the instant the liquid phase vanishes is, according to (16),

$$\langle T\rangle = T_c + \frac{Q}{c}\{\exp(p^2) - 1\}. \tag{19}$$

The corresponding mean value of distribution (18) is

$$\langle T - T_a\rangle = \frac{Q}{c}\sum_{n=1}^{\infty} A_n\frac{\sin\mu_n}{\mu_n}\exp\left(-\mu_n^2\mathrm{Fo}\right). \tag{20}$$

Inserting (16), (19), (18), and (20) into (9) at $\tau = 0$ yields

$$\frac{\sigma_1\left(\frac{x}{l}, \text{Fo}\right)c}{E\alpha Q} = \exp(p^2) - 1 - \sqrt{\pi}\, p \exp(p^2)\left\{\text{erf}(p) - \text{erf}\left(\frac{x}{l}p\right)\right\}$$

$$+\sum_{n=1}^{\infty} A_n\left\{\frac{\sin\mu_n}{\mu_n} - \cos\mu_n\left(1 - \frac{x}{l}\right)\right\}\exp(-\mu_n \text{Fo}). \tag{21}$$

The general distribution of thermal stresses generated at an instant $t > t_e$ is determined, according to the principle of stress superposition, by adding expressions (17) and (21):

$$\frac{\sigma\left(\frac{x}{l}, \text{Fo}\right)c}{\alpha EQ} = \{\exp(p^2) - 1\}\left(1 + \ln\frac{x}{l}\right)$$

$$+\sum_{n=1}^{\infty} A_n\left\{\frac{\sin\mu_n}{\mu_n} - \cos\mu_n\left(1 - \frac{x}{l}\right)\right\}\exp(-\mu_n^2 \text{Fo}). \tag{22}$$

We will now consider an extreme case of this solution, when $p \ll 1$ (low rate of solidification) and $\text{Bi} \ll 1$ (low rate of cooling from the instant $t = t_e$ on). The latter condition means that a change in the original temperature distribution (16) is connected primarily with an equalization of temperatures across the plate section. In this case we have

$$\mu_n = \pi(n-1);\ m_n = \delta_{n1};\ 2h_n\frac{1-\cos\mu_n}{\mu_n^2} = 4\cdot\frac{1-(-1)^{n-1}}{\pi^2(n-1)^2};$$

$$\frac{\sin\mu_n}{\mu_n} - \cos\mu_n\left(1 - \frac{x}{l}\right) = -\cos\pi(n-1)\left(1 - \frac{x}{l}\right),\ n \geqslant 2.$$

Thus, the stress distribution (22) becomes

$$\frac{\sigma\left(\frac{x}{l}, \text{Fo}\right)c}{\alpha QE} = p^2\left\{1 + \ln\frac{x}{l} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{(n-1)^2}\cos\pi(n-1)\right.$$

$$\left.\times\left(1 - \frac{x}{l}\right)\right\}\exp[-\pi^2(n-1)^2 \text{Fo}]. \tag{23}$$

It is easy to see that only the series terms with even subscripts n are nonzero, which contributes to a faster convergence.

In Fig. 2a is shown the distribution of thermal stresses calculated according to expression (23) for solidification at $p = 0.1$ and for various values of the Fourier number. The curve for $\text{Fo} = 0$ has been plotted according to expression (17). A curve representing more severe thermal conditions of plate forming is shown in Fig. 2b. The calculations were based on expression (22) under conditions $p = 0.5$ and $\text{Bi} = 0.5$. The curves correspond to instants of time Fo (0, 1, 2.5, and ∞). The $\text{Fo} = 0$ curve, as before, has been plotted according to expression (17). A comparison between both diagrams of elastic state-of-stress formation in a continuous ingot yields a quantitative indication about the dynamics of stress generation and development under various conditions of plate solidification and cooling. It is easily seen that the diagrams indicate not only different residual stress levels but also different stabilization rates. Thus, in the first case (Fig. 2a) the stress distribution remains close to the residual level and stabilizes within a time $\text{Fo} \cong 0.5$, while in the second case (Fig. 2b) it stabilizes within a time $\text{Fo} \cong 4$. For a completely cooled plate the level and the distribution of residual stresses and strains does not depend, in terms of the elastic-body approximation, on the conditions of ingot cooling after complete solidification. Thus, the general solution (22) yields for the center of the ingot ($x = l$)

$$\frac{\sigma(1, \text{Fo})c}{\alpha EQ} = \exp(p^2) - 1$$

$$+\sum_{n=1}^{\infty} A_n\left(1 - \frac{\text{Bi}}{\mu_n\left(\mu_n^2 + \text{Bi}^2\right)^{\frac{1}{2}}}\right)\exp(-\mu_n^2 \text{Fo}).$$

The residual stresses correspond to Fo → ∞ and approach asymptotically the

$$\frac{\sigma(1,\ \mathrm{Fo})\,c}{\alpha QE} = \exp(p^2) - 1 \tag{24}$$

level.

The effect of the cooling rate on an already solidified ingot becomes the governing factor in regard to crack formation. Namely, as has been said, the rate of external cooling affects the time necessary for temperature equalization. The proper cooling rate should be such as to make the stresses at the thermal center of an ingot increase at the minimum possible rate and, thus, to fulfill the strength requirement in the most dangerous segment of the ingot.

NOTATION

t	is the time coordinate;
τ	is the instant of time when the selected material point moves into the interphase boundary;
t_e	is the end of solidification process;
σ_{ij}	is the stress tensor;
$\dot{\sigma}_{ij}$	is the stress-rate tensor;
ε_{ij}	is the strain tensor;
$\dot{\varepsilon}_{ij}$	is the strain-rate tensor;
Θ	is the one third trace of the strain tensor;
U_{ij}	is the unit tensor;
$\lambda_0,\ \mu_0$	are the Lamé constants;
k	is the modulus of bulk compression;
α	is the linear expansivity;
T	is the instantaneous temperature of solid phase;
T_1	is the temperature of liquid phase;
T_s	is the solidification point;
T_c	is the temperature of the plate surface;
T_a	is the temperature of the cooling medium;
T_0	is the initial temperature of the liquid;
Q_0	is the heat of solidification;
E	is the modulus of elasticity;
ρ	is the density;
$\dot{u}$	is the component of the rate-of-displacement vector;
x	is the space coordinate;
ξ	is the coordinate of the interphase boundary;
l	is half the plate thickness;
λ	is the thermal conductivity;
c	is the specific heat of the solid phase;
c_1	is the specific heat of the liquid phase;
a	is the thermal diffusivity;
p	is the solidification parameter;
α_T	is the heat transfer coefficient;

$\mathrm{Bi} = \alpha_T l/\lambda$;
$\mathrm{Fo} = a(t - t_e)/l^2$.

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